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Postulational Methods. III

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A FEW PROPERTIES OF POSTULATE SETS

WE NOW proceed to define certain terms which we shall apply to sets of axioms and derive a few properties. A great deal of what follows in this section is still based on a two-valued logic, while our criticism of the usual independence and consistency proofs is based on an n-valued logic. The necessary alterations are now being worked out by the author.

Definition. Two postulate sets are said to be prime[1] to each other if no axiom of one of the sets is an axiom or theorem of the other, and conversely, no axiom of the second is an axiom or theorem of the first.

Theorem 1. There exists at least two consistent axiom sets which are prime to each other.

Definition. By *contradictory* propositions we mean two propositions

[1] This term was first applied by H. M. Sheffer to *two postulates*, neither of which implies any part of the other. A set of postulates mutually prime by pairs he calls a set of mutually prime postulates. Cf. Abstract in Bull. Am. Math. Soc., v. 22, 1916, p. 287. We here extend the idea to two sets of postulates. All our results were reached independently, since the abstracts of Prof. Sheffer's paper merely gave the definitions.

p and q such that if p (or q) is *true* q (or p) must be *false*, and if p(or q) is false, q (or p) must be true. q is then denoted as p′ or p as q′.

Proof. Let $A_1 A_2 \ldots A_n$ be one axiom set and $A_1' A_2' \ldots A_n'$ be the other, where A_i' and A_i are the contradictories of each other.

The two sets thus defined are prime to each other, because, suppose they were not and A_j were either an axiom or a theorem of A_i' ($i = 1 \ldots n$). We would then have a) a proposition and its contradictory as axioms which would reduce the set to absurdity; in the sense that anything would follow from it b) a theorem which contradicts an axiom—which would make the set inconsistent. Therefore, the sets are prime to each other.

Definition. If A_i ($i = 1 \ldots n$) is a postulate set, then the set A_i' ($i = 1 \ldots n$) is called the *inverse set.*

Corollary 1. Any axiom set and its inverse are prime to each other.

2. Any axiom set and a subset of its inverse set are prime to each other.

Theorem 2. Any subset of an axiom set and any subset of its inverse are prime to each other.

Proof. Let A_j ($j < n$) be any subset of A_i and A_r' ($r < n$) be any subset of A_i'. We have two cases: (a) $j = r$ (b) $j \neq r$. If $j = r$ (i.e. A_i and A_j') then the two subsets are contradictories and are prime by theorem 1. Suppose $j \neq r$, i.e. A_j contains all those axioms which are not contradictory to the axioms of A_r'. Then if these were not prime, one of the axioms of A_j (or A_r') would be a theorem or an axiom of A_r' (or A_j), and therefore the set A_i ($i = 1 \ldots n$) and A_i' would not be prime. But by theorem 1 they must be.

Definition. By the *logical product* (or briefly product) of axiom sets, we mean the axioms common to both sets. E.g. if $A_1 A_2 \ldots A_n$ is one axiom set, and $B_1 B_2 \ldots B_r B_{r+1} \ldots B_n$ is the second and $A_s = B_r$; $A_t = B_{r+1}$ and for all others A_i, $A_i \neq B_j$ then the product of these two sets is $A_s A_t$ (or $B_r \cdot B_{r+1}$).

Theorem 3. The product of two prime sets is zero.

Proof. This follows at once from the definition of prime sets.

Corollary 1. The product of a set and its inverse is zero.

Proof. Any set and its inverse are prime by Corollary 1, theorem 1.

Theorem 4. If the product of two sets is not zero, then there exists an axiom set, whose theorems are theorems in both the given sets.

Proof. If the product of two sets A_i and B_j is not zero, it follows that these are not prime sets, and therefore there exist axioms common to both sets. Let these axioms be P_r ($r = s, s + 1 \ldots t$). P_r are in A_i and theorems deduced from P_r without use of the set $A_i - P_r$, are

theorems of the set A_i and also of the set B_j because P_r is in the set B_j. Likewise, any theorems proven from P_r without use of the set $B_j - P_r$ are in B_j and in A_i. Furthermore, the theorems deduced from P_r without use of $A_i - P_r$ and those deduced from P_r without use of $B_i - P_r$ are the same theorems being derived merely from P_r.

Theorem 5. A necessary and sufficient condition that two sets have their product zero is that the two sets be prime.

Proof. If the product is zero, then by definition the two sets have no axioms in common, i.e. they are prime. If the sets are prime, then by theorem 3, the product is zero.

Theorem 6. A set and any subset of its prime set are prime.

Proof. We can show this by showing their product to be zero. Let A_i be one set and let B_j be any subset of the prime set B_i'. If any axiom (say B_r) of B_j were in A_i then the product of A_i and B_i' would not be zero, because B_r would be in B_i' also. But by hypothesis B_j' is prime to A_i, therefore A_i and B_j are also prime.

Corollary 1. The product of any set and a subset of a set prime to it is zero.

Theorem 7. If, A_i is prime to B_i and B_i is prime to C_i, then A_i is prime to C_i whenever $C_i = A_i'$ (i.e. C_i is inverse to A_i).

Proof. That A_i is not in general prime to C_i when both are prime to the same set is evident as soon as we take $C_i = A_i$. However, if A_i is prime to B_i and C_i $(= A_i')$ is prime to B_i it follows that A_i and A_i' are prime by definition of A_i'.

Corollary. The relation of "prime to" is not transitive.

Theorem 8. If A_i is prime to B_i, then B_i is prime to A_i $(A_i \neq B_i)$.

Proof. By definition if A_i is prime to B_i, then A_i and B_i have no axioms in common. Therefore B_i and A_i have no axioms in common. Therefore B_i is prime to A_i.

Corollary 1. The relation "prime to" is symmetric.

2. If $A_i = B_i$ the proof fails by definition of "prime." Therefore, the relation "prime to" is not reflexive.

Theorem 9. If the set A_i contains the set B_i as a subset, and if A_i is prime to a set C_i, then B_i is also.

Proof. Suppose B_i were not prime to C_i then B_i and C_i would have at least one axiom in common. But since all axioms of B_i are included among A_i this axiom would be also. Therefore A_i and C_i would not be prime.

Definition. The *sum* of two axiom sets are those axioms of both sets, without those common to the two.

E.g. If $A_1, A_2, \ldots, A_k, A_{k+1}, \ldots A_n$ and $B_1, B_2, \ldots B_k, B_{k+1}, \ldots, B_n$

are the two sets where $A_k = B_k$; $A_{k+1} = B_{k+1}$ then the sum of the two sets will be $A_1, A_2, \ldots A_{k-1}, A_{k+2}, \ldots A_n, B_1, B_2, \ldots B_{k-1}, B_{k+2}, \ldots, B_n$.

Theorem 10. The sum of two sets, neither of which contains an axiom contradictory to an axiom in the other, is an axiom set.

Theorem 11. If the product of two sets A_i and B_i neither of which contains an axiom contradictory to an axiom in the other, is zero, then the sum of the sets A_i and B_i is an axiom set containing all the axioms and theorems of the two sets.

Conversely: If the sum of the two axiom sets is an axiom set containing all the axioms and theorems of the two sets, then the product of the sets is zero.

Theorem 12. If A_i contains B_i as a subset, then the sum of A_i and B_i are the remaining axioms, i.e. $A_i + B_i = A_i - B_i$.

Theorem 13. The product of any set and a subset of itself is the subset. $(A_i \cdot B_i = B_i.)$

Corollary. The product of any set by itself is the set. $(A_i \cdot A_i = A_i)$.

Theorem 14. If A_i contains B_i as a subset, then the sum *of the product of A_i and B_i and the sum of A_i and B_i* is A_i $(A_i \cdot B_i + [A_i + B_i] = A_i)$.

Proof. The sum of A_i and B_i are the remaining axioms of A_i. The product of A_i and B_i is B_i. Therefore, the sum of these two is A_i.

Corollary. If A_i contains B_i as a subset, then the sum and product of these two are prime to each other.

We shall now tentatively define Independence and Consistency for axiom sets and derive some properties of these. Later we shall discuss questions more fully. We first define Independence between two different sets.

Definition. Two sets are said to be *totally independent* if it is impossible to deduce any axiom of the one set from the axioms of the other set.

Theorem 15. There exist at least two sets which are totally independent.

Proof. Take any set A_i and its inverse A_i'. These two sets are the required sets, because suppose some axiom $\begin{Bmatrix} A_j \\ A_j' \end{Bmatrix}$ were deducible from $\begin{Bmatrix} A_i' \\ A_i \end{Bmatrix}$ then $\begin{Bmatrix} A_j \\ A_j' \end{Bmatrix}$ would contradict $\begin{Bmatrix} A_j' \\ A_j \end{Bmatrix}$ and the set would reduce to absurdity.

Theorem 16. Two sets are independent if they are prime.

Proof. Let the sets be A_i and B_i. By definition of "prime" no

axiom or theorem of the one is an axiom or theorem of the other. Therefore the sets are independent.

Corollary 1. Subsets of Independent sets are independent sets.

Proof. Since subsets of prime sets are prime, the corollary follows at once.

Corollary 2. Two sets whose product is zero are independent sets.

Definition. An axiom set is said to be a *self-independent*, or briefly, independent, set,[2] if for any axiom A_c, we can form a set $A'_c + (A_i - A_c)$ such that $(A'_c)(A_i - A_c) \rightarrow 0$ is false.

Theorem 17. If an axiom set is self-independent, then after replacing any axiom by its contradictory, there exists an interpretation for the resulting set.

Proof. An interpretation would be impossible to find only if the resulting set implied an impossibility, i.e. only if $A_{n-r} \cdot A'_r \cdot A_{n-r+2} \rightarrow 0$.

But this implies $A_{n-r} \cdot A_{n-r+2} \rightarrow A_r$. i.e. A_r is not independent of the remaining axioms.

This is the formal proof for the method commonly used to show independence. H. M. Sheffer believes this involves a circle, but the above proof shows no circle. We shall, however, have more to say about this later.

Theorem 18. There exists at least one independent set.

Proof. The set given by H. M. Sheffer for / serves as proof.

Theorem 19. Any subset of an independent set is an independent set.

Proof. If it were not, the complete set would not be independent.

Theorem 20. The product of two sets each independent, is an independent set if the sets are not prime.

Theorem 21. The sum of two totally independent sets, each independent, is an independent set, if no axiom of either set is the contradictory of an axiom in the other set.

Definition. Two sets are said to be *equivalent*, if the axioms of one set are derivable from those of the other, and those of the second are derivable from those of the first.

Theorem 22. There exists at least two equivalent postulate sets.

Proof. The sets for *Principia Mathematica* given by E. V. Huntington in Trans. Am. Math. Soc. 1933.

[2] Sheffer defines an *independent* set as a set of postulates such that if m is the number of postulates no m-1 of them imply the mth. Then he says that a mutually prime set of postulates imply independence. Bull. Am. Math. Soc., 22, 1916, p. 287. We do not develop the properties of primeness. Sheffer's definition is identical with ours.

Theorem 23. The sum and product of two equivalent sets, is either set.

Theorem 24. Two equivalent sets are never totally independent.

Definition. 1. An axiom set is said to be *partially consistent* if when any theorem p is deduced, it is impossible to deduce p'.

2. Two sets are *mutually consistent* if the set which results when we consider the axioms of both sets as one set is a partially consistent set.

Theorem 25. A set and its inverse are never mutually consistent. The proof follows from the definition of two sets mutually consistent.

Theorem 26. The product of two mutually consistent non-prime sets is a partially consistent set.

Proof. Since the product of two sets is a subset of each set, this subset must itself be partially consistent, or each set would not be partially consistent.

Corollary. Any subset of a partially consistent set is a partially consistent set.

Theorem 27. The sum of two mutually consistent sets is not necessarily a partially consistent set.

Proof. Let $A_1A_2 \ldots A_n$ be one set and $B_1A_2'B_3A_4A_5 \ldots B_n$ be the other. The sum of these two sets is $B_1A_2'B_3B_6 \ldots B_nA_1A_2A_3A_6 \ldots A_n$ which is not partially consistent since it contains A_2 and A_2'.

Theorem 28. The sum of two sets which are mutually consistent and equivalent is a partially consistent set.

Theorem 29. The sum of two prime sets is not necessarily a partially consistent set.

Theorem 30. If an axiom set is partially consistent, there exists an interpretation of it.

Proof. If no interpretation existed, it would mean that the set implied an impossibility, i.e. $A_1A_2 \ldots A_n \rightarrow 0$ or $A_1A_2 \ldots A_{n-1} \rightarrow A_n'$

And since any axiom set implies its own axioms $A_1A_2 \ldots A_n \rightarrow A_n$

i.e. the set by definition is not partially consistent.

Corollary 1. If $A_1 \ldots A_n$ is an axiom set, and if when any one of the axioms is replaced by its contradictory $A_1 \ldots A_j' \ldots A_n$ we can find an interpretation then $A_1 \ldots A_j' \ldots A_n$ is a partially consistent set.

2. If an axiom set A_i is an independent set, then the sets formed by replacing each axiom by its contradictory (i.e. $A_1'A_2 \ldots A_n$; $A_1A_2' \ldots A_n$ etc.) are partially consistent sets.

This shows how the problem of consistency can be referred to that of independence and conversely. H. Weyl in *Philosophie der Mathematik* (p. 18) has put the matter in more metaphysical

terms. He says that a scientific theory cannot yield a real contradiction to the given. Therefore, the proof of consistency by representation is a real control. There are, however, certain difficulties involved in both these concepts which we shall discuss later. These are due to the failure to note the distinction between partially consistent and totally consistent sets. Cf. our definition of consistency later.[3]

Although we are quite conscious of the fact that we have used a method of proof which Brouwer has rejected, we have not made use of our analysis of Brouwer for these reasons (a) we are dealing here only with a finite set of axioms; (b) even re-interpreting our results to read "possible" for "true" wherever we have used the law of excluded middle, will still retain all of the properties derived even if in different form.

INDEPENDENCE PROOFS. CRITICISM AND ANALYSIS

We have already shown in *theorem 17* of a previous section that a necessary and sufficient condition for showing independence of a set with one of its postulates, is to find an interpretation of the set with the contradictory of the postulate. In *theorem 17* we only proved part of this, but the converse is easily established. The first attempt at a proof of necessity and sufficiency in verbal form was given by *A. Padoa* at the *International Congress of Philosophy* at Paris in 1900. But Padoa recognize that there were certain difficulties involved when he points out that the failure to prove a system independent proves nothing at all. As a consequence we see that despite the fact that we do have a necessary and sufficient condition, we have no mechanical means for solving the problem directly in any given case. Such a solution will be at hand if we use a method based on the logic of H. B. Smith (which we explain later in this section) and is contingent upon the solution of the general problem of logic referred to previously.

Illustrations of this usual method of showing independence can

[3] For other properties of postulate sets and the definition of further concepts cf. "Axiomensysteme für beliebige Satzsysteme," Teil 1, "Sätze erster grade," Teil II "Sätze höheren grades." Math. Ann. v. 87, 1922, p. 246 ff. and v. 89, 1923, p. 76 ff. This is so far as I know the only other work on this topic.

be found in every article which sets up postulate sets. One such has been given previously and others can be found in articles by Sheffer and Huntington.

Despite the fact that in an interpretation we appear to have a necessary and sufficient condition for independence, a great deal of dissatisfaction has been expressed with this method. Most of it is based on the fact mentioned previously that the failure to find such an interpretation proves nothing. Thus for example, R. M. Eaton says "The Proofs of Independence, consistency and completeness usually employed are thoroughly unsatisfactory; they involved an appeal to interpretations. Suppose that the system has no interpretation, how can the consistency, independence and completeness of the postulates be shown? There should be some analytical way—purely in the realm of the abstract without interpretation—of establishing these properties of a set of postulates." (General Logic 1931, p. 474n). The first part of the quotation shows a misunderstanding. *If it can be shown* that a set has no interpretation, then that set is not consistent, and also another set with an axiom contradictory to one in the original set is not independent. This is due to the fact that this interpretation is a necessary and sufficient condition. But notice, and this is the important fact, that in order to show that a given system is not independent, we must offer an *impossibility proof*, i.e., *a proof that an interpretation is impossible.* Too frequently we hear that such and such a thing has been proven impossible, only to have the impossible accomplished immediately afterwards. As far as possible, we must avoid such proofs and attempt to find purely analytical tests.

The two important objections, then, to this type of independence proof are (1) if no interpretation is found we have proven nothing; (2) to show a system non-independent, we become involved in impossibility proofs.

We are compelled then to consider this necessary and sufficient condition to be either too general or too special. In either case we must find either another condition, both necessary and sufficient or else an equivalent condition, less general or more general as the case may be, which can be applied. We shall indicate the

way to such a condition later after consideration of what has been termed "complete independence." But we cannot deny that an interpretation is a necessary and sufficient condition, since it has been proven so for *ordinary* implication.

When we wish to show a set of axioms $A_1\ A_2....A_n$ consistent, we seek an interpretation for these axioms. When we wish to show A_i independent of $A_1\ A_2....A_{i-1}\ A_{i+1}....A_n$ we replace A_i by A_i' and seek an interpretation of the set $A_1'\ A_2....A_i'...A_n$ i.e. we show this set to be a consistent set. In seeking constants for our variables, we use either concepts of other systems or entities of the physical world. In the first case it is assumed that the system used is itself consistent, and in the second case it is assumed that "reality is consistent." Thus for example, in order to prove Euclidean geometry independent of the parallel axiom, we deny that axiom, and show that the resulting set can be interpreted on a pseudosphere. It is, of course, assumed that since we can do this, we have referred the matter to the consistency of reality. Or again, it was seen that Sheffer referred the independence of his axioms to the consistency of smaller sets which he defined. In either case neither independence nor consistency is proved, but merely referred back to another system. Notice, however, that this criticism does not deny the condition to be necessary and sufficient, but merely points out difficulties in the application of the condition.

Clearly what must be done is to show in a purely analytical fashion that an interpretation exists if *any* consistent system exists.

If we recall certain methods used in the Theory of Infinite Series in mathematics a possible solution, other than that suggested below, is opened to our consideration. We recall that in order to prove that a given series converges or diverges, we compare it with another series known to converge or diverge. This second series was shown to converge or diverge directly. The test is as follows, and can be found in almost any book on Algebra or Series; if a series of numbers is given such that every term is less than every term of a series known to converge, then the given series converges:—if every term is greater than every term of a

series known to diverge then the given series diverges. The important fact to be noted is that certain series are known to converge or diverge by various means.

Now, suppose we have a set of axioms which are finite in number and from which only a finite number of theorems can be drawn. We can show such a system consistent or inconsistent immediately by deducing the complete system. In this way we can obtain systems known to be consistent. If we have two such systems shown to be consistent, we can obtain a greater system by combining these two. The resulting system being finite also can be seen to be consistent or inconsistent by direct investigation.

Given any set of axioms which we wish to show independent, we interpret the variables in terms of one of the systems already shown to be consistent. If the resulting propositions are all true theorems or axioms of the set known to be consistent, then these offer an interpretation and we have shown independence. It is necessary to have as many systems known to be consistent as there are axioms.

For the case where we have an infinite number of propositions to be shown independent we have a much more difficult task and one which can only be solved with the solution of all true and false propositions deducible from the given set. It is true that this case will practically never occur, but it must be considered if we are to be able to say our solution is general. At any rate, it is a possible case unless it can be shown that no infinite set of propositions can be an independent set. If we mean the infinite set of all propositions, (which is one of the paradoxical sets) then clearly that set cannot be an independent set (i.e. if it exists abstractly it cannot be) since it contains propositions and all propositions deduced from them. There are then different infinite sets of propositions. Abstractly considered, the set defined by the property that no one of its elements can be deduced from any one or any combination of the others, is a possible set.

In order to show such a set to be an independent set we must either show that *any* axiom of the set except the one in question, satisfies the interpretation or that *any* sub-set of the set is an independent set. In the first case we note the structure of the

axioms and show that they must satisfy the interpretation. In the second case we show that *any* sub-set is independent and further that any two such sub-sets are mutually independent and therefore their combination will be an independent set as proven above.

In most cases independence has not really been solved, but the question passed from one system to another. Each system set up to prove independence for each particular axiom is assumed consistent. Very frequently a circle does arise. The set A is shown consistent by reference to system B which is itself shown consistent with reference to system A.

It is obvious in how close a connection independence and consistency are. O. Holder offers an excellent illustration for the case of the parallel axiom, of how independence is proven, by reference to the consistency of Lobatschefski's non-Euclidean Geometry. The problem of the Independence of the parallel axion of Euclid, is the most famous problem of all. As is well known many attempts were made to deduce the axiom—since it did not appear to be as self-evident as it should. Not the least of these attempts is Saccheri's "Euclides Vindicatus" and in the introduction to the translation made by G. B. Halsted it is remarked "Saccheri was the first to use this procedure (i.e. of construction of examples) and constructs his example without leaving the field of formal logic" (Open Court 1920 p. XXI).

However, it appears upon closer examination that Saccheri's attempt was related rather to the inconsistent triad than to an actual interpretation as discussed just above. This method is actually the method of indirect reduction which is denoted by the letter "k" in the mnemonic names *Baroko* and *Bokardo* of the Aristotelian logic. The name antilogism was given to this by Mrs. Ladd-Franklin. Briefly the procedure is that if a given proposition is implied by other propositions, to deny it and combine it with some of the other propositions should lead to a contradiction of a known truth. As an example, consider the mood Baroko. This is AOO in the second i.e. A(ab) O(cb) implies O(ca). In this we assume that A(ab) and O(cb) are to be taken true, i.e. they might be considered as the axioms of the system. Now the

question is, does O(ca) follow as a theorem from these two? If it does not, i.e. if it is independent of A(ab), and O(cb), then A(ab) O(cb) should imply A(ca). But if we consider A(ab) and the theorem A(ca) together we have a mood in Barbara implying A(cb). But A(cb) contradicts O(cb) and since in our implication A(ab) A(ca) implies A(cb) A(ab) was given true, it follows that A(ca) cannot be true, i.e. to assume O(ca) is independent of A(ab) and O(cb) leads to a contradictory system. Therefore, O(ca) is not independent of the others. As an illustration of this method to other propositions one can use the following three propositions:

1. All men are mortal
2. All lawyers are men
3. All lawyers are mortal

The application of this method to independence proofs is obvious from the discussion. The objection which occurs is again the clumsiness of the method. For the case (which is rare) in which we *can* deduce all the theorems of the system with the contradictory of the axiom in question this method is applicable. But, as in the case of Saccheri, where this is not possible we only have a probable consistent resultant system unless we can show the contradiction at once.

But in the last analysis this method is merely another unwieldy form of our necessary and sufficient condition as stated above. In its general form the inconsistent triad is as follows:

If p q r ... implies T, then by contradiction and interchange p q r...T′ implies a contradiction (where T′ indicates the contradictory of T), and that is the condition stated previously.

Symbolically the law is as follows: $pq \rightarrow r: \rightarrow : pr' \rightarrow q'$ which is really contradiction and interchange and can be derived using

1. $p \rightarrow q \cdot \rightarrow p' + q$
2. $p + q \equiv q + p$
3. $(pq)' \equiv p' + q'$

Proof.

$$pq \rightarrow r \cdot \rightarrow \cdot (pq)' + r \text{ (by 1)}$$
$$\cdot \rightarrow \cdot p' + q' + r \text{ (by 3)}$$

$$\cdot\rightarrow\cdot p' + r + q' \text{ (by 2)}$$
$$\cdot\rightarrow\cdot (pr')' + q' \text{ (by 3)}$$
$$:\rightarrow: pr' \rightarrow q' \text{ (by 1)}$$

taking one step further, we have.

$$pq\cdot\rightarrow\cdot r:\rightarrow: pr'q \rightarrow o$$

In the preceding discussion no attempt was made to analyze more closely the definition of Independence. In any set of propositions there are involved a set of concepts, a set of relations affirmed to exist between these concepts and a set of properties. As a result, we must consider briefly what may be called "Intensional Independence." If two concepts are entirely unrelated then, of course, the two propositions which involve them must be independent. If it can on the other hand be shown that two concepts are derivative, one from the other, then the propositions which affirm or rather enunciate their properties must be derivative, and hence non-independent. For example, if we accept Whitehead and Russell's doctrine that the concept of implication is definable in terms of disjunction, i.e. implication can be derived from disjunction, then any theorem relating to implication must be derived or derivable from those of disjunction. In fact many of the theorems of *Principia* about implication are proven by reducing the implicative relations to disjunction. As a consequence, implication is non-independent of disjunction. Only one of these two can be taken as a primitive idea. Two axioms are independent if a new concept occurs in one which is not in the other,—independent in the sense that one cannot be derived from the other.[4] On the basis of this it follows that two propositions, one of which contains a concept contradictory to a concept in the other, are independent, but not of course consistent.

Taking the set of elements in any set of propositions, we can have the following cases:—1) the set of propositions affirm of the same concepts,[5] the same relations; 2) they affirm of the same

[4] Cf. Weyl "Handbuch der Phil. der Math," p. 18.

[5] We mean by concept here the subjectival concept as opposed to the adjectival and relational concept.

concepts different relations; 3) they affirm of the same concepts the same properties; 4) they affirm of the same concepts different properties; 5) they affirm of different concepts the same relations; 6) they affirm of different concepts different relations; 7) they affirm of different concepts the same properties; 8) they affirm of different concepts different properties.

Clearly in the cases 1) and 3) we do not have independent propositions. In so far as the same concepts must have the same properties 4) is impossible, if however two or more propositions affirm various properties individually, they are equivalent to one proposition. The remaining cases must be separately considered.

To affirm of the same concepts different relations gives two cases a) the relations are independent; b) the relations are derivative. In b) we have the case discussed above for implication, and the propositions are therefore non-independent. In a) we have what are called "external" relations and the propositions are independent. As an illustration: given the concepts a, b, and the relations $\sim$ and $+$, then the propositions

$$\sim\sim \mathrm{a} = \mathrm{a}$$
$$\mathrm{a} + \mathrm{b} = \mathrm{b} + \mathrm{a}$$

are independent. This takes care of 2).

5) Is seen to involve dependence immediately, since by generalizing the propositions we can derive all the given propositions. As an example; given the different concepts *a*, *b*, *c*, *d* and the relations R, R′, and *a* R *b*, *a* R′ *b*, *c* R *d*, *c* R′ *d*, we can by using variables x, y write: given the concepts *a*, *b*, *c*, *d* and the relations xRy; xR′y. The four propositions then become theorems.

6) Involves independence immediately.

7) Means either the properties enunciated are not all the properties, or else the concepts are not different. If the properties are not completely stated, it follows that the concepts are derivative under limitations.

8) Involves independence immediately.

To set up a logic of independence then based upon intensional factors, would involve a consideration of the eight cases men-

tioned above. We do not intend, however, to undertake that task here, since we are concerned with purely formal independence, which is a prior question.

Although we cannot say that if two concepts can be attributed to the same object they are derivative, we can say that if two concepts cannot be attributed to the same object they must be incompatible. Hence it follows that given two concepts, if we can attribute them to the same object, they are compatible. It follows, further, that in the ordinary proof of independence by interpretation we demonstrate at the same time the compatibility of the concepts involved.

The underlying assumption in interpretations which makes an appeal to the experiential world is that contradictory attributes cannot exist in the same objects. It follows from this assumption that if we can give such an interpretation of our propositions we have been successful, because the propositions involved are compatible or consistent. This assumes the validity of the law of contradiction, and of course involves the metaphysical doctrine of Hegel "The real is the rational."

Returning now to our definition of independence, we note two cases:—1) a proposition p is independent of a proposition q if p cannot be deduced from q and q cannot be deduced from p. Both aspects must be taken into consideration, since we would not call a proposition p independent of q if p implied q but q did not imply p. The other two possible cases are cases of dependence also. 2) where we have a set of propositions, $A_1 A_2 \ldots A_n$ no A_i must imply any other A_j ($i \neq j$ nor must any combination of any number of propositions imply any one of the propositions.[6] The terms irredundant, irreducible, and completely independent have also been used for independence. In view of the fact that these two cases are possible, it is advisable to use the term Irredundant[7] for the second case.

Material Implication must, as has frequently been asserted, be ruled out of consideration. If we allow material implication

[6] Cf. Lewis and Langford *Symbolic Logic*, p. 337.

[7] Cf. A. Church, *On Irredundant Sets of Postulates*. Trans. Am. Math. Soc., 1925, p. 318 ff.

the whole problem becomes meaningless,[8] since any proposition is implied by a false proposition, and a true proposition is implied by any proposition. Thus as Professor Lewis points out, two propositions can never be independent and consistent at the same time. But from what has been said, our entire concern is to find a set of propositions which are both independent and consistent. We are faced then with the alternatives: either we deny the validity of material implication, or admit it doesn't apply in this case. It is quite possible that in postulate sets we are concerned not with implication, but with *deducibility*, but are we then to admit that if we are dealing with a set of propositions (not necessarily axioms) we cannot speak of their independence and consistency? It becomes clear that logic is overlooking certain distinctions which must be made.

In consideration of the second case mentioned above we are faced with the possibility that although in a set of postulates we may have independence in pairs, the set may not be completely independent. This seems to have been recognized by E. H. Moore. In order to solve this problem Moore makes certain remarks which will lead us later to a solution.[9]

Any proposition p divides the universe into two compartments —p and p′ (not −p). Two propositions p and q would then divide the universe into the following compartments: pq, p′q, pq′, p′q′

[8] In an article in the Monist (v. 29, p. 152 ff.) Lenzen shows that the only set of postulates which are independent is one of *only one* postulate. The proof is based on the assumption that $q \cdot\!\!\rightarrow\cdot\ p \rightarrow q$ (2.02 of *Principia*). He reduces Huntington's sets of independent postulates to non-independence as follows for A_2 and M_5

$$A_2\ (a + b) + c = a + (b + c)$$
$$M_5\ a \times b = b \times a$$

Proof

$$a \times b = b \times a :\rightarrow: p \cdot\!\!\rightarrow\cdot\ a \times b = b \times a$$
$$\therefore p \cdot\!\!\rightarrow\cdot\ a \times b = b \times a$$
$$\therefore (a + b) + c = a + (b + c) :\rightarrow: a \times b = b \times a$$

This can obviously be done for all postulates in any set.

[9] Cf. E. V. Huntington's discussion in *Postulates for Betweenness*, Trans. Am. Math. Soc., 26, 1924, p. 277.

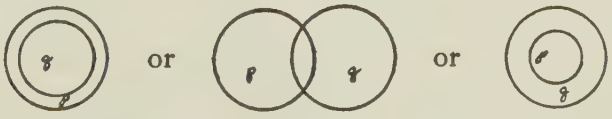
(a)

	p	p′
q	pq	p′q
q	pq′	p′q′

In the case of three propositions p, q and r, we have pqr, pq′r, pqr′, pq′r′, p′qr, p′qr′, p′q′r, p′q′r′. These can be represented by intersecting circles. Suppose all of these regions exist, then there is no relation of implication between them. If any one does not exist, any pqr′ → o, then we have pq → r, or pq and r are not completely independent. In general, we can say that if we have n propositions, we shall have 2^n compartments. If any one of these 2^n compartments do not exist, there is a relation between the propositions involved. This gives rise to the doctrine known as the "complete existential theory."

It is to be noticed that if any region in the universe represents an impossibility, we have non-independence, since the region o.pq′ → o i.e. an impossibility implies anything. This is different, however, from saying that a false proposition implies any proposition; since p′q′ is not necessarily zero. This shows clearly how material implication which fails to make necessary modal distinctions must lead to strange results. If falsity = impossibility, then what Lewis says about independence and consistency in this type of implication holds. But Brouwer and Smith have clearly seen that this is not so—modal distinctions must be made.

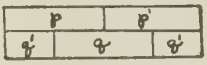
The method outlined is, of course, another approach to the necessary and sufficient condition for independence given previously. It expresses in diagrammatic form the same condition. *This is not the same as ordinary implication.* If p implies q, the relation can be represented as



(a)

as will be seen below. However, with the introduction of other propositions, we have always the possibility that these spheres will intersect and allow of implication. It ought, however, to be possible to reduce the latter case to the former. E. G. if $pq'r'$ does not exist, then of course $pq' \rightarrow r$ i.e. the intersection of pq' should contain r or be contained by r. At any rate, we ought to come out the same way.

If we represent the four compartments as we have done above, it is possible that we are overlooking a very important case. Suppose we use a form we shall use later:

(b) 

where these horizontal bars are to be considered to lie one atop the other. We notice that $(pq \rightarrow o)'$, $(p'q' \rightarrow o)'$, $(pq' \rightarrow o)'$ and $(qp' \rightarrow o)'$—yet there is an implicative relation of some sort between p and q; between p and q' between p' and q' since they do overlap. In other words:—if pq' actually vanish, (i.e. $pq' \rightarrow o$ then $p \rightarrow q$ because q contained all the p, since that is the only way $pq' \rightarrow o$. By reference to the diagram (a) above, this will seem to be so. By reference, however to diagram (b) it is evident that if pq' vanishes, then p is the q. With the addition of more propositions the situation becomes more and more complicated, and it becomes necessary to find a reduction formula. It is evident that the method outlined by Huntington for complete independence is not valid, since it considers only a special type of diagram, i.e. a special case of overlapping. The method as stated, can still be used for consistency despite its speciality. It is necessary only to demonstrate the existence of a compartment, to show the consistency of the propositions whose intersection this represents.

This discussion substantiates our previous remarks that proving a set pq.....r′ to have an interpretation shows the consistency of pq....r′ and therefore the independence of pq.....r. The failure to do so tells us nothing. *We must conclude, therefore, that the condition for independence which was said to be necessary and sufficient is so only for a special type of implication, namely that in*

which p is in q or q is in p. In the case of diagram (b) if $pq' \rightarrow o$ then p does imply q *but if* $p \rightarrow q$ *it does not therefore follow that* $pq' \rightarrow o$. Some negation of q with p vanishes, but this question is not to be discussed here. It appears that certain other modal distinctions are necessary which have been overlooked. We have shown that when (b) occurs, we do not have independence for all meanings—i.e. there may be some meanings for which p does imply q.

About a year after Huntington's article cited above, a paper appeared in the Transactions of the American Mathematical Society[10] by A. Church entitled "On Irredundant Sets of Postulates." This seems to be an attempt to remedy the situation just described by shrinking some of the postulates until total inclusion or exclusion results. We outline this paper in part.

In classic logic we know that if e.g. A implies I, that A is said to be a *strengthened* form of I and I a *weakened* form of A. If we have *any valid* implication then the antecedent is the strengthened form while the consequent is the weakened form. This distinction Church adopts when he says that if p implies q then q is said to be weaker than p. Given a set of postulates $A_1\ A_2 \ldots\ldots A_n$ then A_1 can be weakened with respect to the set A_i ($i = 1 \ldots\ldots n$) if there exists a postulate A_k, weaker than A_1 such that the logical product of $A_k\ A_2 \ldots A_n$ implies A_1.

A set of postulates is then defined to be *irredundant* if the postulates are independent and no one of them can be weakened with respect to the set.

We see immediately that if a proposition A_i can be weakened with respect to the set, then that set cannot be an independent set, i.e. at least one axiom can be derived as a theorem. Suppose A_1 implies A_k and the set $A_1\ A_2 \ldots\ldots A_n$ is weakened to $A_k\ A_2 \ldots A_n$. This means that $A_k\ A_2 \ldots A_n$ implies A_1. But A_1 implies A_k Therefore $A_k\ A_2 \ldots A_n$ implies A_k. Therefore A_k can be derived. from a theorem and is therefore a theorem.

Church then goes on to indicate the relationship between *complete independence* of Moore and this relation of Irredundancy.

[10] V. 27, 1925, p. 318 ff.

If we represent the set of all possible mathematical systems by a square, a postulate is then an inner square which divides the square into two regions, one satisfying the postulate, the other not.

The shaded square represents the region of sets satisfying the postulate. Call this region B.

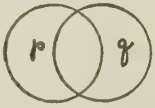
Then if we have a postulate B such that A is weaker than B, A would lie within this larger region B. This is the ordinary idea of implication.

We proceed to our own analysis.

Before doing so we wish to state that proceeding as we must on the basis of a logic still not general, at least with respect to negation, our methods here are only tentative.

If two propositions are in dependence, they may be represented in two ways

1.

2. 

Case 2 shows those instances for which a proposition implies another only for *certain* values. Case 2 may be represented as:

i.e., p may intersect q in more than one region. However, this reduces to Case 2 as follows; we can either break up p into two parts p_1 and p_2 having no common values, and retain both, as e.g.

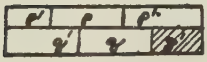
or else we can throw away one part of p as already contained in q, namely that part of p which is in q on one side or the other, as e.g.

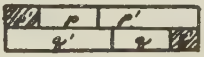
It is not necessary that the regions be well-behaved. They are represented in the way they are merely for convenience,—the important thing being their intersections.

Having reduced intersections in this way, we shall represent them in a different fashion as follows[11]

Case 1

Case 2


Case 3

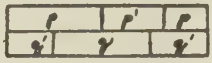

Case 4

The upper horizontal line is to be conceived as lying entirely on the lower, and the line is to be considered the universe. Thus in case 1, q is entirely within p. In the case of more than two postulates we merely add another line above these, taking into account all the possible combinations and making the reductions indicated in case of a repetition of intersections. Thus the shaded parts in case 2, 3, and 4 can be reduced as described above.

Case 4 is the only case that can properly be called independence. In all other cases the propositions are related in such fashion that at least for one value there is an implicative relation. We call case 4, total independence; case 3, partial independence; cases 1 and 2 non-independence. In cases 3 and 4, it is noticed that pq′, p′q, (case 3 only) and p′q′ are none zero. In case 1, p′q is zero, and in case 2, pq′ is zero.

[11] Cf. H. B. Smith *Abstract Logic or the Science of Modality*, Jan. 1934 (privately distributed manuscript).

This suggests the possibility of the following case which is also



Case 5

not a case of independence.

We tabulate our results

Case 1 $p'q = 0$
Case 2 $pq' = 0$
Case 3 none zero
Case 4 $pq = 0$
Case 5 $p'q' = 0$

As long, then, as we have the condition that p and q may overlap even in part, we cannot speak of total independence. E. H. Moore's *Complete Independence* and A. Church's *Irredundant Sets* are only cases of partial independence, and hence their methods must be inadequate. They base their systems, as have all attempts to define independence, on the incomplete disjunction "either Case 4 or Case 2 are alone possible." As a consequence, if Case 4 is shown to be invalid, Case 2 is left. It is now evident that Case 4 may be invalid, and we may *not* have Case 2. This is really the heart of the argument we directed against the methods outlined above.

The argument for independence was as follows:—if p implies q, then $pq' = 0$. Therefore if $pq' \neq 0$ p does not imply q. Applying this to Cases 2 and 4, it holds. But in Case 3 for example, $pq' \neq 0$, and for *some* values of p, p does imply q. Independence proofs as we said before are therefore invalid and new ones must be developed.

Now given a set of postulates $A_1 \ldots A_2$, we shall say they are totally independent only if we have

i.e. their spheres do not overlap at all. There are no values which satisfy any one which also satisfy any other. As a result, in order to prove a set totally independent, we need only prove the postulates so independent in pairs. (We recognize that this involves certain difficulties with respect to consistency—the solution of this will be given below.) In order, however, to draw inferences from the product of any combination, it will be necessary to weaken the postulates with respect to the set. Having shown that the postulates are totally independent in pairs, we can weaken the postulates and get a set independent in the usual sense. We must, however, take care not to weaken any postulate so much that its new form embraces entirely any of the other postulates. The converse operation is also possible. Having given a set which has been shown to be independent in the sense that a system exists satisfying all but one of the postulates, we can then go about strengthening the postulates until we have a totally independent set. If this cannot be done, it is because at least one of the postulates lies wholly within one of the others, i.e. is not independent at all.

We can state the condition (a) for *total* independence as follows: if $A_1\ A_2 \ldots\ldots A_n$ are the postulates, then $\pi\ A_i\ A_j = o\ (i \neq j)$ $(i, j = 1 \ldots n)$ (π indicates the logical product)

(b) for *partial* independence if A_i implies A_{k_i} then under same conditions

$$\pi\ A_j\ A'_{ki} = o\ (A'_{ki} \text{ means contradictory of } A_{ki})$$

We would be compelled at this point to end the discussion and await the complete development of Logic were it not that Smith's system of Logic is well on its way to completion and offers us a sign-post of the direction of advance. As a consequence we shall indicate certain portions of his Logic and show a formula which gives us so far as we know a method for the solution of this problem. Its application will be given in connection with the problem of the next chapter. It must be remarked that the complete use of this formula involves a knowledge of the atomic elements of Logic as well as those of the system to which it is applied.

The general problem of Logic is to set up all true and false

propositions involving any number of implication symbols. In order to do this it is necessary to have a means of reducing any complex of propositions to a sum of products of modal functions. This formula Smith has given as

1. $| pq | = pq + p'q' | p || q | + | p || q' | ' + | q || p' | '$.

However it was shown that this does not hold in certain cases. It is precisely for those cases that we do not need this formula as a condition for independence.

We define

Def. 1. $| p | = (p \to o)'$ which may be read as *p is possible*; $p \to o$ is to mean that *p is impossible*. $(p \to o)' =$ *it is false that p is impossible*. $| p |' = p \to o$ read *p is impossible*.

By $| p |$ we mean that there exist values which satisfy p.

$|| p ||$ $= [(p \to o)' \to o]'$ and may be read *it is possible that p is possible*.

p by itself means *p is true*.

Postulate $p \to (p \to o)'$ i.e. $p \to | p |$

p′ means there exists at least *one* value such that p is not satisfied. This is to be distinguished from $p \to o$ which means that there are *no* values for which p is satisfied.

As a consequence p′ is consistent with $(p \to o)'$. Thus the formula can be read as follows

p and q are possible together means either p is true and q is true, *or* p is false but possible, and q is false but possible, *or* p is possible and it is impossible that q be false, or *q* is possible and it is impossible that p be false.

Example. I and O are possible together means *either* I and O are true together, *or* I is false but possible and O is false but possible, *or* I is possible and it is impossible that E be true, *or* O is possible and it is impossible that E be true.

By contradicting 1, we can obtain after reduction

2. $| pq |' = | p |' + | q |' + p q' | p' | + qp' | q' |$

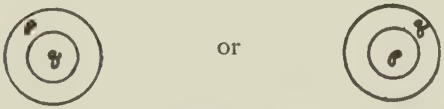
the expansion for p and q being impossible together.

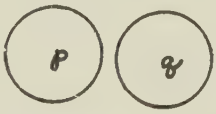
Recalling now that we defined *partial independence* as meaning

the existence of values in common and *total independence* as the existence of no values in common, we can interpret 1 and 2 as follows.

In order for two propositions to be independent *totally* they must be impossible together, i.e. | p q | ′ must evaluate to *one*, or | pq | must vanish. Therefore given two propositions (say A and E) if when we substitute in 1, the resulting form vanishes, then we say that *A* and *E* are impossible together, i.e. A is completely independent of E. Note that we are not using precisely the same concepts that Smith uses—although perhaps they amount to the same in the end.

These formulas have been shown to break down for the cases where p and q are related as

(a) 

(b) 

i.e. when the formulas fail then we have cases (a) or (b), but the converse is not true, when we have cases (a) or (b) the formulas do not always break down. As a result, we can say that when given two propositions p and q, the formulas fail to be either zero or one, then we have a case of ordinary implication, (i.e. non-independence) or total independence. Other considerations would then be necessary to determine which of the two cases we have. But we feel that this breakdown is due to the application of the formula to ordinary implication. If we interpret the matter in terms of independence, we have a temporary tool which can be used in many cases. But even for the case of implication, the formulas can be handled comfortably even though necessitating some amount of work, as Smith shows in "Abstract Logic." (Cf. below where a general method of independence is given on the basis of the above formulas.)

CHAPTER 8. CONSISTENCY PROOFS. CRITICISM AND ANALYSIS

In a lecture held at Zurich in 1932, P. Bernays indicated three types of consistency proofs.[12] These were 1) The methods of evaluation 2) The methods of "Ausintegrierens" (as illustrated in the methods of Lowenheim and Skolem) and 3) The methods of Eliminations. 1) Is the method used by Hilbert and illustrated in chapter 1.[13] This is weakened because of its dependence upon a reference to other systems assumed to be consistent. 3) Is really another form of 1) and hence reduced the three methods to two.

A great many of the criticisms we directed against independence proofs hold for consistency proofs. Most of the usual definitions and proofs of consistency for a set of postulates depend upon the consistency of some other sphere. The usual definition of consistency has been that two or more propositions are consistent, if there exist values which satisfy them all simultaneously. Again if we accept the consistency of Reality we are forced to argue in this fashion:—the Real is consistent, therefore that which can be attributed to the Real *under similar circumstances* must be consistent. Hence if given a set of propositions we can interpret them in terms of Reality, they must be consistent. However, since Logic expresses or attempts to express relations as formal as possible, it must be the aim of Logic to find, as for the case of Independence, purely formal conditions for consistency. We should be able given any propositions or *propositional functions*, to determine from their form alone, whether or not they are consistent.

It may be objected, that despite formal consistency the concepts involved may be inconsistent.[14] This, however, overlooks a very important fact.

In general the concept appears to determine the form of the proposition into which it enters. If, for example, our concept

[12] Cf. Verhandlungen des Intern. Math. Kongresses Zurich, 1932, p. 342–343.

[13] Another good illustration is in Veblen and Young's "Projective Geometry," p. 2–3. We do not reproduce it here.

[14] Carnap in the Abriss der Logistik sets up two columns, one for concepts and one for propositions (p. 70). We show this to be unnecessary.

is a class-concept, the form of the proposition into which it enters will be of class inclusion or exclusion—i.e. A, E, I, O, U, V; which Smith has shown in his work to be necessary and sufficient to express the relation of classes to each other. If, the concept is a relational one, then the proposition into which it enters will be of the form xRy. Now if we have given two concepts which are contradictory, and desire to use them the following will happen. We express these concepts in propositional form and use these propositions. Obviously, we do not use concepts except in such form. Since these concepts determine the form of the propositions they will either 1) determine forms obviously contradictory or 2) determine forms contradictory but not obviously so.

Another argument is as follows. Concepts in and by themselves, apart from any other concepts, are neither consistent nor inconsistent with these other concepts. It is only when these concepts are related in some way to other concepts, that they can contradict. But concepts are related to other concepts only in propositional form. Therefore, if the concepts are contradictory, the propositions which express the relations between these concepts will be contradictory. As a consequence, it follows that all we need show is formal consistency or inconsistency, since this includes "intensional" consistency.

Of course, we must mention that it may happen that the concepts are consistent while the propositions they enter into are not. This case is very easily recognized and reduces immediately to formal inconsistency. It is therefore not important to our considerations.

Despite the fact that consistency is a very vital property of a postulate set, very little attempt has been made to solve the problem in general. For the most part only the properties of the ordinary methods were investigated. Thus for Carnap a consistent set of axioms is a "fulfilled" set, i.e. one which has a model (Erkenntnis Vol. 1, p. 302); and for Hilbert existence = consistency; while for the Intuitionists consistency is existence, but existence means constructability. As far back as 1771 Lambert in "Anlage zur Architektonik" (p. 16) had already said "The possible is what contains no contradictions." In all these cases

the consistent set of propositions has an interpretation or model. Having such an interpretation, it is argued that it will then be impossible to deduce a proposition p and the contradictory of that proposition at the same time. Here again the assumption is that if a set of properties can be attributed to a system that system is an exhibition that the properties are consistent and as a consequence, since contradictory attributes cannot belong to the same object, we can never get contradictories from the set of properties. This method, depending as it does upon finding a model, must therefore be considered weak.

Since the definition of consistency involves "implication"—e.g. pqr ..... are consistent if when pq .... imply x, it is impossible that pq ..... imply x′—the close relationship between this problem and Logic becomes evident. In fact, it is an amazing situation that this has not been recognized earlier. Fraenkel[15] suggests that the problem must be referred to Logic ultimately. He feels this to be a very difficult task because of a possible circularity, due to the fact that it will necessitate a proof of the consistency of Logic, which he claims is not less problematic than mathematics. However, granting an absolute proof of consistency possible, such a proof must use logical principles, i.e. it must assume a system of logic already consistent. So we seem to be led either to a circle or to a vicious infinite regress. This again overlooks the nature of Logic.

Suppose we were asked to define a "square." Upon saying that a square is a plane figure having four sides, the question is raised, "Is this definition 'square'?" Obviously a nonsensical problem. The real problem to be raised is—*Is this definition a definition of "squares?"* And this can only be answered by applying it to the objects supposed to possess that property. So Logic is, in its general form, a *definition of consistency*, and therefore it is meaningless to ask whether Logic is consistent. The way to answer the problem—*is this a definition of consistency?* is to apply it to those objects supposed to have that property—namely, sets

[15] Cf. Einleit in die Mengenlehre, p. 365.

of propositions. As a consequence, if given a set of variable propositions, we can interpret them in logical terms or attain a logical model, we do have a consistent set of propositions. Or else we apply a method to be given below which determines the implications between propositions under certain conditions. It is to be noted that the definition of consistency

$$p \text{ cons. } q \equiv [(p \rightarrow q)\ (p \rightarrow q')]' [p \neq o]$$

is itself a logical principle. But note below the definition we use which avoids the difficulties in this.

To the above two objections might be raised: 1) Logic itself is in a very doubtful state: 2) the objection of Fraenkel that Logic uses Number-theory and Theory of Aggregates, and cannot therefore be used as a consistency proof without a circle. To the first objection we answer, that all that signifies is that a consistency proof would still be impossible if Logic were not developed far enough, since we would not have all the conditions of consistency. To the second we answer, that such an objection is untenable. Logic does use the concept of class and of number, but these are logical concepts fundamentally and are therefore defined in Logic or taken as the undefined elements of Logic. Mathematics merely develops these concepts in their application to other objects. There is, therefore, no circle involved. Furthermore, although it is true that Mathematics is not the particular of which Logic is the general, yet in a very real sense Mathematics does and must use the principles of Logic. Although there are common factors, it is not correct to say that Logic involves certain branches of mathematics as mathematics involves Logic. The principle of the Syllogism in Logic merely is the method of obtaining consistent propositions.

Here again all the remarks made concerning material implication and Independence hold for material implication and consistency. In other words, the application of material implication to these problems brings into clear light the lack of generality of material implication. It prevents a set of propositions from being both consistent and independent, yet we have many sys-

tems which have both these properties. But it will be recalled that we defined total independence as propositions related as

where there is no overlapping, yet in order to have consistency there *must be* overlapping. Are we then to infer that totally independent propositions cannot be consistent?

It is necessary to distinguish two cases a) contradictory propositions b) non-contradictory propositions. If a) is true then obviously they cannot be consistent. In such a case total independence implies inconsistency. However, if the propositions are not contradictory, but merely refer to distinctly separated entities, such need not be the case. We use the concept of weakening, to define consistency for totally independent propositions.

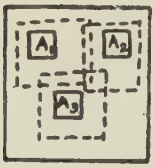
Let $A_1\ A_2 \ldots A_n$ be a set of propositions which are totally independent. Then if the propositions can be so weakened that they do have a common area, we call the set of totally independent propositions, *consistent with respect to total independence.*

For example, given a set of three propositions $A_1\ A_2\ A_3$ totally independent. This is illustrated by

Now if $A_1 \rightarrow \bar{A}_1$; $A_2 \rightarrow \bar{A}_2$; $A_3 \rightarrow \bar{A}_3$ where

$\bar{A}_1\ \bar{A}_2\ \bar{A}_3$ imply A_1
imply A_2
imply A_3

then if we denote the widened area with broken lines, we have



when we have consistency. If this is not possible (as in the case of contradictory propositions) we say the set is *inconsistent with respect to total independence.*

It is to be noted that in consistency, the logical product of any two propositions must not be irremediably zero. In independence we have two cases a) the product of any two is zero, but the product of one by the strengthened form of the other, or the product of the strengthened forms of both must not be zero b) the product of any one with the contradictory of the other must not be zero. We can express these conditions symbolically thus:

Let π stand for "logical product of"
A' stand for contradictory of A
$\bar{A}$ be the weakened form of A

a) Total Independence

$$\pi A_i A_j = o \ (i \neq j) \ (i, j = 1,2....n)$$

b) Partial Independence

$$\pi A_i A_j' \neq o \text{ (same conditions)}$$

c) Consistency with respect to total Independence

$$[\pi A_i A_j = o] \ [\pi A_i \bar{A}_j \neq o]$$

d) Ordinary consistency

$$\pi A_i A_j \neq o$$

For case c we can also have

$$[\pi A_i A_j = o] \ [\pi \bar{A}_i \bar{A}_j \neq o]$$

For cases b) and d) it is necessary to add

b′) $\pi A_1 A_2 \ldots\ldots A_j' \ldots\ldots A_n \neq o \ (j = 1,2\ldots.n)$

d′) $\pi A_1 A_2 \ldots\ldots A_n \neq o$

For cases a) and c) cases similar to b′) and d′) are unnecessary, since to show the conditions to hold in pairs shows the condition to be true for the product of all.

If we interpret $= o$, as meaning "is impossible," then the cases show the existence or non-existence necessary for independence and consistency. As a consequence, we try to prove certain theorems.

Theorem 1. A set of propositions are consistent if there exists a range of objects satisfying them.

Proof. Suppose there was not. Then if $A_1 \ldots A_n$ is the set

$$A_1 \ldots A_n = o \text{ by definition of } = o$$

or

$$A_1 \ldots A_{n-1} \rightarrow A_n'$$

and therefore

$$A_1 \ldots A_{n-1}$$

cannot be consistent with A_n, because suppose it were, then by definition $A_1 \ldots A_{n-1} A_n \neq o$ for some values at least. But by hypothesis this is impossible.

Theorem 2. If a set of propositions have a range of objects satisfying them, they are consistent.

Proof. If there is a sphere it means that $A_1A_2 \ldots A_n \neq o$ for some values at least, and therefore no subset will imply the contradictory of a proposition in the subset not containing the given subset for these values at least.

Therefore $A_1A_2 \ldots A_n$ is consistent for these values.

The above two theorems constitute a necessary and sufficient condition for consistency. However, all our objections hold, and it is necessary as for independence to find a more or less general condition. If postulates are incompatible, no model can be found for them. Unless an impossibility proof is set up, the failure to find a model can give us no solution.

It is necessary, therefore, to prove two other theorems:

Theorem 3. If a set of propositions are consistent we can find a model in a finite number of steps.

This theorem is by no means evident, and may be impossible of proof when the so-called *Transfinite Axioms* are involved. If it can be proven then, we know that *if* we have a consistent set, we will eventually reach a model. We must then show

Theorem 4. If a set of propositions are inconsistent, we can show in a finite number of steps that a model is impossible.

The converse of theorem 3 is obviously true. The converse of theorem 4 is also obvious. It is the theorems as they stand which appear to be impossible of proof—particularly 4. And furthermore, if we throw 3 and 4 into negative forms as

Theorem 3'. If we cannot find a model in a finite number of steps, then the set is inconsistent.

Theorem 4'. If we cannot show in a finite number of steps that a model is impossible, then the set is consistent

we see that proving 3 and 4 will be extremely difficult. Theorem 4' is untrue because the failure to find a model in a finite number of steps, as was said before, proves nothing.

This immediately suggests a possible method of consistency proof by induction. This method consists in showing that the set of postulates is consistent for at least one value, then assuming it consistent for another value, to show that it must be consistent for the next. But if the set is consistent for *one* value, then we need proceed no further. It is a consistent set. No one claims, I believe, that in order for a set to be consistent it must be consistent for all possible values. Another variation of this method would be to show say two postulates to be consistent in a set of n postulates, then to assume $r < n$ consistent, and show that then $r + 1$ would be. It is, however, difficult to see how we could prove that if r propositions are consistent, that $r + 1$ would then be unless the $(r + 1)$ is always a theorem from the r. It does not follow that if one sub-set of a set of propositions is a consistent set, that the whole set is, any more than it follows that if one

subset of a set of numbers is ordered that the set is an ordered set. It would be necessary to show that *any* subset no matter how selected, is a consistent set. But if this can be shown, the induction is unnecessary, since it follows at once that the set is consistent.

As interesting as all these attempts are, it must be said that they are premature. Consistency depends for its solution upon the completion of logic, and upon a knowledge of the atoms of any given system whose consistency is to be shown. Meanwhile efforts should have been directed to discover the properties of consistency, in this way furthering, really, the advance of logic. Lewis and Langford have attempted to do this in "Symbolic Logic." We shall discuss this system elsewhere. Meanwhile we give an extension of his definition which we feel is necessary.

We shall use to express the fact that p and q are consistent;

Definition. $\Box\, pq = (p \rightarrow q')' + (p' \rightarrow q)'$

From which we get

$$- \Box\, pq :\equiv: [p \rightarrow q']\ [p' \rightarrow q] :\equiv: p = q'$$

We use the ordinary set of postulates for Logic.

$\Box\, pq$ is to be read *p is consistent with q*

$- \Box\, pq$ is to be read *p is not consistent with q*

$(p \rightarrow q)'$ is to be read *p does not imply q*

p' is to be read *p is false* or *not — p*

We set up our truth-value matrices.

p	q	$(p \rightarrow q')'$	$(p' \rightarrow q)'$	$\Box pq$
T	T	T	F	T
T	F	F	F	F
F	T	F	F	F
F	F	F	T	T

This shows that under our definition, two true propositions and two false propositions can alone be consistent, never a true and a false. This takes care of consistent lying.

Notice that according to our definition, the consistency of p and q means that either it is false that p and q together are impos-

sible or else that p′ and q′ together are impossible. If we consider the universe to be divided into 4 quadrants by our two propositions, then our definition affirms the existence of the two quadrants pq and p′q′. To affirm the existence of only one of the quadrants is not sufficient. We give below a further method of analysing the results that follow upon the assumption of the existence or non-existence of the various quadrants. Meanwhile we prove:

Theorem 1. $\Box p = (p \to o)' + (o \to p)'$

Proof. In definition let $q = 1$

$\Box p \cdot 1 = \Box p \equiv (p \to o)' + (p' \to 1)' \equiv (p \to o)' + (o \to p)'$

Theorem 2. $\Box pq = \Box qp$

Proof. $\Box pq = (p \to q')' + (p' \to q)'$

$= (q \to p')' + (q' \to p)'$

$= \Box qp$

Theorem 3. pq′ is the usual definition of Independence.

Proof. $\Box pq' \equiv (p \to q)' + (p' \to q')'$

$\equiv (p \to q)' + (q \to p)'$

i.e. p does not imply q or q does not imply p.

Now if it is possible to have two propositions both consistent and independent, the two relations $\Box pq$ and $\Box pq'$ must be possible for some values at least. By definition $\Box pq$ is not the contradictory of $\Box pq'$ (the contradictory being $- \Box pq$)

Theorem 4. $(\Box pq)(\Box pq') \gtrless o$

Proof. $\Box pq = (p \to q')' + (p' \to q)'$

$\Box pq' = (p \to q)' + (q \to p)$

$(\Box pq)(\Box pq') = (p \to q)'(p \to q')' + (p \to q)'(p' \to q)'$

$+ (q \to p)'(q \to p')' + (q \to p)'(q' \to p)'$

If any one of the terms is not zero, the sum cannot be.

Let us take $(p \to q)'(p \to q')'$ and rewrite it as $(pq' \to o)'$. $(pq \to o)'$ which we interpret as meaning pq′ does not vanish and also pq does not vanish.

Using a diagram we used before

P′	p	P′
q′	q	q′

which shows that both pq and pq′ exist. This clearly shows that p is consistent with q′ is not the same as p is inconsistent with q.

Theorem 5. $p \rightarrow \Box\, p$

Proof. $p(p = o) \rightarrow o$

$p = o \equiv (p \rightarrow o)(o \rightarrow p)$ Df.

$p(p \rightarrow o)(o \rightarrow p) \cdot \rightarrow \cdot o$

$\Box\, p = (p \rightarrow o)' + (o \rightarrow p)'$ by theorem 1

$\therefore - \Box\, p = (p \rightarrow o)(o \rightarrow p) \equiv (p = o)$

$p(- \Box\, p) \rightarrow o$

$p \rightarrow \Box\, p$

i.e. if p is true, it is self-consistent.

Theorem 6. $p \rightarrow p(- \Box - \Box\, p)$

Proof. $- \Box\, p \cdot \equiv \cdot p = o$ by definition

$- \Box - \Box\, p \equiv \Box\, (p = o)$

$= [p = o] \rightarrow o + o \rightarrow (p = o)$

$p[\Box - \Box\, p] = p[(p = o) \rightarrow o] + \cdot p[o \rightarrow (p = o)]$

Using now $p \rightarrow q \cdot \rightarrow \cdot p' + q$ we obtain

$$p[(p = o)' + o] + p[1 + (p = o)] = p(p = o)' + p(p = o) + p = p$$

$p \Box - \Box\, p \rightarrow p$

or

$$p \rightarrow p(- \Box - \Box\, p)$$

i.e. p is true implies that p is true and at the same time its inconsistency is inconsistent.

This is very much like the famous Brouwer axiom.

Theorem 7. $\Box\, (pq)r = \Box\, p(qr)$

Proof. $\Box\, (pq)r = (pq \rightarrow r')' + [(pq)' \rightarrow r]'$

$\Box\, p(qr) = [p \rightarrow (qr)']' + (p' \rightarrow qr)'$

We show the terms equal thus

$[p \rightarrow (qr)']' = (pqr \rightarrow o)' = (pq \rightarrow r')'$

$(p' \rightarrow qr) = (o \rightarrow pqr)' = [(pq)' \rightarrow r]'$

$\therefore \Box\, (pq)r = \Box\, p(qr)$

This theorem shows that if any *n* propositions are consistent, then any n-r combinations will be consistent with the other r.

We notice that in terms of the symbolism used by Dr. Smith our definition of Consistency can be expressed as follows:

$$\Box\, pq = |\, pq\, | + |\, p'q'\, |$$

which suggests a use of the expansion for | pq | as given above.

Furthermore, it is to be noticed that consistency always can be interpreted in terms of the vanishing of certain quadrants. In the case of two quadrants, if (p q′) or (q p′) vanish, always, then the propositions are not independent, since then $p \rightarrow q$ in the other case $q \rightarrow p$. If the quadrants (pq) or (p′q′) vanish, then the propositions are inconsistent. There always remains the final case in which none of the quadrants vanish. Before offering a method of *quadrant analysis*, we remark that the process can always be generalized even though a very lengthy process may be involved, due to the complexity of the forms of the given propositions. As a result what holds of two propositions in this analysis may be extended into a general case. It can easily be seen for the case of n-propositions that if the segments representing n-1 and the contradictory of the nth vanish, that the propositions are not independent. If the segments representing the n propositions and the contradictory of the n propositions vanish, then the propositions are inconsistent. This may be due to the fact that some subset are inconsistent, but that we can obtain another subset which is consistent. It becomes necessary as a result, to discard propositions one by one until we get a consistent set. If we never do reach such a consistent subset, it may be because the propositions are totally independent, and it becomes necessary to weaken the propositions. If this cannot be done so that a consistent set is reached, then we are certain that the propositions are inconsistent.

We now offer one method of testing for quadrant analysis.[16] We explain only the cases of two propositions and three propositions, because on the basis of Theorem 7, we can continually reduce to these instances.

We have defined

a) $\quad | p | = (p \rightarrow o)'$ therefore we obtain
b) $\quad | pq | = (pq \rightarrow o)'$

[16] A great deal of what follows is due to Prof. H. B. Smith. It is advisable to consult his works before reading what follows. Cf. especially "Abstract Logic."

As special cases we obtain

a′) $|\,o\,| = o$
b′) $|\,1\,| = 1$

For convenience we rewrite the expansion formulas

c) $|\,pq\,| = pq + p'q'\,|\,p\,|\,|\,q\,| + |\,p\,|\,|\,q'\,|' + |\,q\,|\,|\,p'\,|'$
c′) $|\,pq\,|' = |\,p\,|' + |\,q\,|' + pq'\,|\,p'|' + qp'\,|\,q'\,|$

We recall that c) and c′) fail whenever one of the quadrants vanishes. Thus when the formulas are applied and they break down, we are certain that one at least of the quadrants vanish.

The best way to explain the analysis is to take a concrete example. Therefore let

1) $p = (a \to b')' + (b' \to a)'\,\{(a \to b) + (b \to a)\}$
$q = (a \to b)' + (b \to a)'\,\{(a \to b') + (b' \to a)$

Conditions: 1) p and q are functions of the propositions a and b whose forms are given

2) $a \to b \equiv$ a implies b

Problem: To determine under what conditions on the propositions entering into the functions, the functions p and q are independent or consistent; i.e. what conditions on a and b will make the various quadrants of p and q vanish or remain.

We first re-write 1) and 2) in modal symbolism

1′) $p = |\,ab\,| + |\,b'a'\,|\,\{\,|\,ab'\,|' + |\,ba'\,|'\}$
2′) $q = |\,ab'\,| + '\,ba'\,|\,\{\,|\,ab\,|' + |\,b'a'\,|'\}$

We now assume ab = o (i.e. ab are inconsistent together) while the other quadrants exist. This is the following case.

a′	a	a′
b	b′	b

3) Then $p = |\,o\,| + |\,b'a'\,|\,\{\,|\,ab'\,|' + |\,ba'\,|'\}$

Since a′b′, ab′, and ba′ do not vanish we can apply the formulas c and c′).

$$| b'a' | = a'b' + | a' | | b' | + | b' | \cdot | a |'$$
$$| ab' |' = | a |' \text{ since } a \rightarrow b'$$
$$| ba' |' = | a'b |' = | b |' \text{ since } b \rightarrow a'$$

3′) $$p = | a | '| b' | + | a' | | b |'$$

using as theorems: $- | o | = o$

$$a' | a |'b' \rightarrow | a |' \cdot b'$$
$$a | a |' = o \text{ and } o.x = o$$
$$b' + | b' | = | b' |$$
$$| a |' \cdot | a |' = | a |'$$
$$| a |' \cdot | a' | = o$$

3″) $$p = \{ | a | + | b' |'\} \{ | a '|' + | b |\}$$
$$= | a' |' + | a | | b | + | b' |' | a' |' + | b' |'$$

or $$p' = | a' |' + | a | | b | + | b' |'$$
$$q = | a | + | b | \{1 + | b'a' |'\}$$

4) $$q = | a | + | b | \text{ using as theorems}$$
$$| ab' | = | a |$$
$$| ba' | = | b |$$
$$1 + x = 1$$

4′) $$q' = | a |' | b |'$$

We are now ready to analyze our quadrants:—

$$pq = \{ | a |' | b |' + | a' | | b |'\} \{ | a | + | b |\}$$

5) $$pq = | a | | a' | | b|' + | a|' | b'| | b| = o \text{ since } a \rightarrow b'; b \rightarrow a'$$

$$p'q = \{ | a' |' + | a | | b | + | b' |'\} \{ | a | + | b |\}$$

6) $$p'q = \{ | a' |' + | a | | b | + | b' |'\} = p'$$

$$pq' = \{| a |' | b' | + | a' | | b |'\} | a |' \cdot | b |'$$

7) $$pq' = | a |' | b |' + | a|' | b |' = | a |' | b|' = q'$$

8) $$p'q' = \{ | a' |' + | a | | b | + | b' |'\} | a |' \cdot | b |' = o$$

Using as theorems for

5) $$| a | | a |' = o$$
6) $$| a | \cdot | a' |' = | a' |'$$
7) $$| b| '\cdot | b' | = | b |' \text{ and } x + x = x$$
8) $$| a |' \cdot | a' |' = o$$

We shall analyze now the case in which none of the quadrants vanish; namely

a′	a	a′
b		b′

For the case above, p and q were consistent and not independent, while p′ and q′ were inconsistent.

In this case our expansion formulas hold, and we must apply them.

9) $pq = [\,|\,ab\,|\,+\,|\,b'a'\,|\,\{\,|\,ab'\,|'\,+\,|\,ba'\,|'\}]$
$[\,|\,ab'\,|\,+\,|\,ba'\,|\,\{\,|\,ab\,|'\,+\,|\,b'a'\,|'$

which can easily be verified not to vanish since in order to vanish each term of the result must vanish and the product term of $|\,ab\,|\,|\,ab'\,|$ reduces to $|a'\,|'\,b\,|\,b'\,|$.

The remaining cases are merely matters of detail. They can easily be verified by actual multiplication in every case.

It is significant to note in conclusion that this method gives not only the data for pq and p′q′ but for all possible cases. In other words, the method tells much more than ordinary independence and consistency facts.

APPENDIX I

Consistency Proof for Converse of Brouwer's Axiom with Lewis' Postulate Set

In the appendix to Lewis and Langford's "Symbolic Logic" is given a proof of the consistency of the A set of axioms. Later in the appendix certain other axioms are introduced—in particular the Brouwer axiom, C 12. It is further stated that this axiom is consistent with the set A and the usual consistency proof is given.

That we can no longer place the usual faith in consistency proofs alone is evidenced by the following consistency proof of the A set with the *converse* of Brouwer's axiom. This would not be so interesting were it not for the fact that the Brouwer axiom is also independent of the other axioms. It was not thought necessary to prove independence of the converse because if it were not, then we should have a theorem

$p = \sim \Diamond \sim \Diamond p$ breaking down all distinctions between a strict and material system.

Our proof shows of course that

$p = \sim \Diamond \sim \Diamond p$ is also consistent with the other axioms.

What is needed is a proof of the generality of a set of axioms.

Consistency Proof

A 1. $pq \rightarrow qp$
A 2. $qp \rightarrow p$
A 3. $p \rightarrow pp$
A 4. $p(qr) \rightarrow q(pr)$
A 5. $p \rightarrow \sim (\sim p)$
A 6. $p \rightarrow q \cdot q \rightarrow r :\rightarrow : p \rightarrow r$
A 7. $\sim \Diamond p \rightarrow \sim p$
A 8. $p \rightarrow q \cdot \rightarrow \cdot \sim \Diamond q \rightarrow \sim \Diamond p$
A 9. $\sim \Diamond \sim \Diamond p \rightarrow p$ (this is the converse of the Brouwer axiom)

We use the matrix for pq and that for p as in the text, but alter that for p in one respect only.

pq	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

p	$\sim p$
1	4
2	3
3	2
4	1

p	$\Diamond p$
1	1
2	1
3	2
4	3

We define $p \rightarrow q$ as meaning p includes q; where any number includes itself and all numbers less than itself.

The main implication sign in 6 and 8 holds if and only if the inclusion on the right is true.

Proof. 1. Postulates 1–6 are consistent since they do not involve $\Diamond p$ where the alteration was made.

2. Postulate 7.
$\sim\Diamond p\rightarrow\sim p$

p	$\sim\Diamond p$	$\sim p$
1	4	4
2	4	3
3	3	2
4	2	1

3. Postulate 8.
$p\rightarrow q:\rightarrow:\sim\Diamond q\rightarrow\sim\Diamond p$

p	q	$\sim\Diamond q$	$\sim\Diamond p$
1	1	4	4
2	1	4	4
3	1	4	3
4	1	4	2
2	2	4	4
3	2	4	3
4	2	4	2
3	3	3	3
4	3	3	2
4	4	2	2

4. Postulate 9.
Brouwer-converse.
$\sim\Diamond\sim\Diamond p\rightarrow p$

$\sim\Diamond\sim\Diamond p$	p
2	1
2	2
3	3
4	4

That a consistency proof is in itself not sufficient can be shown by the following argument.

Let us assume we have a set of propositions $P_1, P_2 \ldots P_n$ which have been shown to be consistent. But let us further assume that p_n is a proposition obtained from $\bar{p}$ by mere specializing of the variables i.e. the propositions p_i $(i = 1 \ldots n - 1)$ do *not* enter into the deduction of p_n. It is further assumed that $p_1, p_2, \ldots p_{n-1}$ are also consistent.

Now consider—having written down our set $p_1 \ldots p_n$, and not knowing that

$$p' \text{ implies } p_n$$

we can show that $p_1 \ldots p_n$ is a consistent system.

But notice that we will have only a special set of axioms and not a general one.

Independence proofs will not help us here because if p′ is independent

of the others p_n will also be, since it is obtained by mere specialization of the variables in p'.

This shows the fact that consistency alone is not sufficient for axiom sets and that a consistency proof does not in the last analysis prove very much. This is further supported by the fact that despite the proof of consistency for *Principia Mathematica*, Huntington's sixth set of postulates contains a postulate not in *Principia Mathematica* and independent of the axioms of *Principia Mathematica*. This fact has been brought out by B. A. Bernstein in a paper in the Bulletin of the American Mathematical Society for October 1933. (Huntington's sets of Postulates are in Trans. Am. Math. Soc. 35 pp 274–304.)

			p	→	~	◇	~	◇	p
p	~	◇	p		~	◇	~	◇	p
1			4						2
2			4						2
3			3						3
4			1						4

shows the Brouwer axiom is independent of the set.

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